Faustmann’s Formulae for Forest Capital

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Abstract. The relationship of the economics of a simple, stylized forestry to capital theory is studied. Investment and two r-percent rules are discussed. A forest’s two resources, the stand and the land, act in conjunction as a single capital asset. If capital is comprehensively defined, there is no role for the concept of an internal rate of return. A forest provides real options in optimal and sub-optimal rotation patterns. Old growth has superficial similarities to a exhaustible resource, but the forest still consists of two resources that, in conjunction, behave comparably to a plantation forest.

Key words: comprehensive investment, r-percent rules, internal rate of return, real options, old growth

JEL Classification: Q23
1. INTRODUCTION

Historically, a stylized plantation forest has been a main vehicle for developing the theory of capital. A remarkable analysis was provided in 1849 by Martin Faustmann (1968). The contribution of forestry to economics, and vice versa, is a theme in Gane’s (1968) examination of papers by Faustmann and Faustmann’s contemporary, E.F. v. Gehren.

Each of two dissimilar equations is sometimes called Faustmann’s formula by students of forest economics. What may be called Faustmann’s formulae formalize the broad considerations involved in the harvest decision. They provide a link, through the prevailing rate of interest, of the forest to other assets and the rest of the economy.

The present paper uses Faustmann’s model to shine light on some questions in capital theory. One of the formulae applies to both optimal and non-optimal forest management and affords a simple comparison of the two. It is possible to consider the implications of an insightful forester’s recognizing that maximum present value should be pursued rather than, for example, maximum sustainable yield, strongly criticized by Samuelson (1976). The discussion also stresses that a forest is a single asset made up of two natural resources, namely (1) the growing trees or stand and (2) the land, comprising land area, soil, climate and general capacity to grow a stand. Both resources contribute in concert, and not individually, to the realization of value. They remain a composite even in the special situation of old growth.

Faustmann’s two formulae, which are the foundation for the present paper, are well known. Some reinterpretations are presented as Theorems 1 and 2. A number of further results follow from the formulae and are presented herein as corollaries to the theorems.
2. TWO RESOURCES, ONE ASSET

Consider a forest under stationary conditions and under certainty. Because conditions are stationary, the age of harvest must be the same in each rotation. Also, the interest rate, \( r \), is taken to be constant through time.

Faustmann only briefly mentions the choice of rotation age. His main preoccupation is with a forest for which, in modern terminology, a particular resource-allocation mechanism is in place (Dasgupta and Mäler 2000). In this case the mechanism predetermines a rotation age. That age is not necessarily optimal. Let it be represented by \( a > 0 \), the net revenues of the harvest by \( R(a) \) and the planting cost by \( c \).

A forest may also provide flows of thinnings, amenity values and costs, \( \alpha_i(t) \), \( i = 1, \ldots, n \), at any time \( t \in (0, a) \) (Faustmann 1968, Hartman 1976, Strang 1983, Pearce 1994). Let the set of indices of internalized flows, identified as amenities or simply flows hereinafter, be denoted by \( I \subseteq \{1, 2, \ldots, n\} \). A flow of cost, for maintenance or administration, is a negative value. Only the algebraic sum of the internalized flows is used below. Therefore, let \( \alpha(t) = \sum_{i \in I} \alpha_i(t) \). Also let \( \int_0^t \alpha(s) e^{-rs} ds = A(t) \) denote the present value of internalized flows up to time \( t \leq a \). Let \( F(a) \) be defined to be the sum of revenues from harvesting at age \( a \) and the cumulated, internalized values of the flows:

\[
F(a) = R(a) + e^{ra} A(a).
\]

Most writers, including Faustmann, have concerned themselves with determining the value of the land. If the rotation age is predetermined to be \( a \), the present value of bare forest land is

\[
L(a) = -c + \frac{F(a)}{e^{ra}} - \frac{e^{ra}}{e^{2ra}} - \frac{F(a)}{e^{2ra}} - \frac{e^{2ra}}{e^{2ra}} + \ldots \\
= -c + \frac{F(a)}{e^{ra}} + \frac{1}{e^{ra}} \left[ -c + \frac{F(a)}{e^{ra}} - \frac{e^{ra}}{e^{2ra}} + \frac{F(a)}{e^{2ra}} - \frac{e^{2ra}}{e^{2ra}} + \ldots \right] \\
= -c + \frac{F(a)}{e^{ra}} + \frac{1}{e^{ra}} L(a). \tag{1}
\]
The following were derived by Faustmann (1968: 30):

\[ L(a) = \frac{F(a) - ce^ra}{e^{ra} - 1}; \]  

(2)

\[ L(a) = -c + \frac{F(a) - c}{e^{ra} - 1}. \]  

(3)

If \( L(a) < 0 \), or equivalently if \( F(a) - ce^{ra} < 0 \), for every \( a \) then the forest is not planted. In an interesting problem there is an \( a_0 \) such that \( F(a_0) - ce^{ra_0} = 0 \), and \( a > a_0 \) for a forest that is planted. For the rotation age \( a_0 \) the value of the land, \( L(a_0) \), is zero. This minimum rotation is depicted in Fig. 1.

Equation (3) can be rearranged to reveal analytic features of the problem beyond solving for the value of the land. By equation (3), for any fixed rotation age \( a > a_0 \),

\[ L(a) + c = \frac{F(a) - c}{e^{ra} - 1} \]

and

\[ L(a) + F(a) = F(a) - c + \frac{F(a) - c}{e^{ra} - 1} \]

\[ = e^{ra} \frac{F(a) - c}{e^{ra} - 1} \]

\[ = e^{ra} (L(a) + c). \]  

(4)

During a rotation, the value of the land is not recovered. At the end of the rotation, it is returned intact, along with the harvested timber. Formula (4) expresses the following \( r \)-percent growth rule.

**Theorem 1** Faustmann’s Formula No. 1. *Let the resource-allocation mechanism set a predetermined, repeated time of harvest or rotation age that is not necessarily optimal. The sum of the value of bare land and the planting cost grows at the rate of interest to the sum of the values of (a) the harvest, (b) the cumulated, internalized amenities and costs and (c) the bare land.*
An implication of formula (4) is that investment in the forest includes the value of bare land in addition to the planting cost. Even though it has no alternative use, bare land has an opportunity cost: there is a choice of the planting time. The bare land is an asset committed at time \( t = 0 \) to the enterprise over the rotation period.

**Corollary 2** Investment. *The investment when land is bare is the sum of the planting cost and the value of the land for the chosen rotation period.*

The \( r \)-percent growth rule (4) is not explicit in Faustmann’s analysis or in other analyses such as Samuelson’s (1976) or Hartman’s (1976). Given the chosen rotation period the total investment earns a rate of return of \( r \). The rule implies that, for a forest, once investments and yields are comprehensively defined to include the land, the internal rate of return is redundant.

**Corollary 3** Internal rate of return. *(i) The internal rate of return on the total value invested in a forest, including the value of the land, is equal to the rate of interest. (ii) A finding that the internal rate of return is greater than the rate of interest is indication that forest capital has not been defined comprehensively.*

The distinction between discounted cash flow and the internal rate of return, discussed inconclusively by Gane (1968), is unnecessary. The value of the forest is discounted into the land value, \( L(a) \), according to equation (1), no matter what the value of \( a \), and \([c + L(a)]\) always grows to \( F(a) + L(a)\). When all investments (planting and land) are taken into account, the internal rate is the market rate. The internal rate is used as a criterion of investment by not considering the forest land to be invested, as if it is not scarce (has no value). Conventionally, the internal rate is given by the solution, \( \rho \), to the equation \( c(1 + \rho)^a = F(a) \). Maximizing the internal rate of return gives a different value from maximizing present value.

Faustmann and Samuelson define the land rental to be \( rL(a) \). For any rental no greater than this value, the rotation \( a \) can be supported with a non-negative profit or
rent accruing to the forester. Even if there is a rental contract, the value \( c + L(a) \) is invested through the forester’s choice and is paid back with interest in return for the use of the land. Computation of the full rental, \( rL(a) \), presupposes computation of \( L(a) \). The essence of the analysis is the role of the land and stand as capital, and not how the forest is financed. Fig. 1 depicts the value of a forest with chosen rotation (resource-allocation mechanism) \( a_1 \), given all of its contributions.

Faustmann (1968) insists that the land value remains \( L(a) \) at all times. The land would be worth \( L(a) \) if it were bare. But it is not bare. During a rotation the land is not available for sale independently. Faustmann’s contemporary, Gehren (1968: 24), comes closer to realizing the point: “At the time of planting, the acre becomes forest.” On the other hand, Gehren wishes to assign a portion of the total value to the land.

The interplay of land and stand is fundamental to formula (4). Once invested, the land and stand form a composite, a forest. The value of the land can be realized only by cutting the stand. The value of the stand can be realized only by restoring the bare land.

**Corollary 4** A forest with standing trees is a single asset that comprises two natural resources, the stand and the land.

The land is transformed to a forest by planting. The land and the planting cost remain invested until the harvest time \( a \). At time \( a \) the forest is transformed to (i) the harvest with value \( R(a) \) and (ii) bare land with value \( L(a) \).

Amenities accrue at rate \( \alpha(t) \). The cumulated value of the flows of amenities is a part of what is returned, in addition to the value of the harvest. The flows are attributable to the forest and not to either natural resource alone.

To stress the comprehensiveness of the forest, let the total value invested be written

\[
\Phi(a) = c + L(a) .
\]
According to formula (4), the total investment $\Phi(a)$ grows at rate $r$ to the total value obtained at the end of the rotation (including the cumulated value of flows of amenities and costs):

$$\Phi(a) e^{ra} = [R(a) + A(a) e^{ra}] + L(a) = F(a) + L(a).$$

### 3. OPTIMAL MANAGEMENT

The optimal rotation age, $\hat{a}$, maximizes the net present value (discounted cash flow) of the forest:

$$L(\hat{a}) = \max_a \frac{F(a) - ce^{ra}}{e^{ra} - 1} > 0.$$  \hspace{1cm} (5)

Setting $L'(\hat{a}) = 0$ yields that

$$F'(\hat{a}) = \frac{F(\hat{a}) - c}{e^{\hat{a}r} - 1} r e^{\hat{a}r};$$  \hspace{1cm} (6)

$$F'(\hat{a}) = r (F(\hat{a}) + L(\hat{a})).$$  \hspace{1cm} (7)

Sometimes equation (7) is considered to be Faustmann’s formula. Faustmann (1968) makes allusions to the possibility of doing better than following the resource-allocation mechanism that is in place. But he does not solve for the optimal rotation.

Another way to express equation (7) is to let $V(a) = F(a) + L(a)$. Since $L'(\hat{a}) = 0$,

$$V'(\hat{a}) = F'(\hat{a}) + L'(\hat{a}) = F'(\hat{a}) = r (F(\hat{a}) + L(\hat{a})) = rV(\hat{a}).$$ \hspace{1cm} (8)

Formula (8) is the expression of another r-percent growth rule (cf. Cairns and Davis 2007: 470).

**Theorem 5** Faustmann’s Formula No. 2. At the optimal strike point, the total, cumulated, internalized value of the forest (land value plus current internalized value) is growing at the rate of interest.

This exact r-percent rule is depicted in Fig. 2 as the tangency at $a = \hat{a}$. It is perhaps noteworthy that, if forest land is not scarce, so that $L(a) = 0$ for all $a$, then Irving
Fisher’s formula for optimal rotation-by-rotation exploitation is an immediate special case.

**Corollary 6** Fisher’s formula. If \( L(\hat{a}) = 0 \) then

\[
F'(\hat{a}) = rF(\hat{a}).
\]

Equation (6) is equivalent to Hartman’s equation (9) or (10). However, Hartman (1976: 57) expresses his r-percent rule as a rule for the increase in stumpage value and formulates it using a “correction factor” to the rate of interest. The interest rate in this partial-equilibrium model is a parameter taken by the forester. It is a required return, earned on the entire investment. A rate of interest, even corrected, applied to the stand neglects the investment of the land and is appropriate only when Corollary 6 holds.

**Remark 1** An “adjusted” rate of return that exceeds the market rate is an indication that forest capital is not being defined comprehensively.

Formula (8) states that at the optimal strike point, value, appropriately defined, rises at exactly the rate of interest. Consider the expression \( F'(t) / [F(t) + L(\hat{a})] \). Since \( L'(\hat{a}) = 0 \), the expression is the rate of change of the cumulated internalized value (from immediate cutting) of the forest. It is equal to \( r \) at \( \hat{a} \), and is a decreasing function at \( \hat{a} \). (See the appendix.) A general result found by Davis and Cairns (2011) applies to the forest.

**Corollary 7** The rate of increase of cumulated internalized value is greater than the rate of interest before and less than the rate of interest after the optimal harvest age.

The optimal rental for this forest is \( rL(\hat{a}) \); it leaves no rent in the hands of the forester. If an annual rental is charged that is less than \( L(\hat{a}) \), there is no change to the optimal rotation period.
A change in a stumpage charge (for harvested timber) does affect the rotation by affecting the value of $F(a)$. The solution in equation (8) is subtle because it depends on a rate of growth. Both the rate of growth and level of $F(a) + L(a)$ are affected by a shift of $F(a)$. As has been found by many authors, since

$$\frac{1}{F(a) + L(a)} \frac{dF(a)}{da} \bigg|_{a=\hat{a}} < \frac{1}{F(a)} \frac{dF(a)}{da} \bigg|_{a<\hat{a}},$$

an explicit consideration of the land as a resource shortens the rotation as compared to considering only the stand as a resource, with the harvest time being optimized rotation by rotation as in Fisher’s formula.

One amenity may be to fix carbon at rate $\alpha_C(t)$, $t < a$ (cf. Pearse 1994). At times $t > a$ the harvested wood may be burnt or decay and return carbon to the atmosphere at rate $\beta_C(t, a)$. (See Kooten, Binkley and Delcourt 1995; Ariste and Lasserre 2001; Cairns and Lasserre 2004, 2006.) Consistently with the assumption of stationarity, let carbon have a constant price $p$ in the permit market. Then the present value of the net contribution of carbon of the forest for rotation period $a$ is

$$AC(a) = \int_0^a p\alpha_C(t) e^{-rt}dt - \int_a^\infty p\beta_C(t, a) e^{-rt}dt.$$

Since $\int_0^a \alpha_C(t) dt = \int_a^\infty \beta_C(t, a) dt$, it is easily seen that, because of “the power of compound interest” $AC(a) > 0$: The contribution of a single rotation of a stationary forest to mitigating climate change is positive, even though all of the carbon is eventually returned to the atmosphere.

If there are only two sources of value, the harvest value and the value of fixing carbon, then

$$\frac{d}{da} \left( \frac{F(\hat{a}) + L(\hat{a})}{F(\hat{a}) + L(\hat{a})} \right) = \frac{d}{da} \left( \frac{R(\hat{a}) + L(\hat{a})}{R(\hat{a}) + L(\hat{a})} \right) + \frac{d}{da} \left[ e^{ra}AC(\hat{a}) \right].$$

At the harvest age $a_C$ that would be optimal if the carbon credits were neglected,

$$\frac{d}{da} \left[ e^{ra}AC(a_C) \right] = r + \frac{p \left[ \alpha_C(a_C) + \beta_C(a_C, a_C) \right]}{AC(a_C)} > r = \frac{\frac{d}{da} \left( R(a_C) + L(a_C) \right)}{R(a_C) + L(a_C)}.$$
Therefore,
\[
\frac{d}{da} \left( F(a, C) + L(a, C) \right) / \left( F(a, C) + L(a, C) \right) > r;
\]
under stationary conditions, an internalization of the value of carbon fixing increases the optimal harvest age.

4. OPTION

Faustmann (1968) chastises Gehren (1968) for evaluating an “immature” stand (one of age less than the harvest age under the prevailing resource-allocation mechanism) at its immediately realizable value rather than discounting the value that can be anticipated if the forest grows to maturity. In the present notation, the distinction is between \( R(t) + L(a) \), \( t < a \) and the value,

\[
[F(a) + L(a)] e^{-r(a-t)} - A(t) e^{rt} = [c + L(a) - A(t)] e^{rt},
\]
that remains in the forest (that has not been received by time \( t \) through the flows).

Gehren (1968) did not even consider the possibility of deviating from his specified rotation age of eighty years, and as a result came to a contradiction that he failed to resolve. Faustmann (1968) pointed out that eighty years was too long, that 65 years was a more profitable choice. But he did not take the step of finding or even discussing the optimal rotation.

The value that remains is the value of the immature forest: As is noted by Samuelson (1976), if the forest is sold as a whole the seller can realize \([c + L(a) - A(t)] e^{rt}\). If the two resources are sold piecemeal, \( R(t) + L(a) \) can be realized. Is the latter greater than the former? Is the whole greater than the sum of the parts?

Consider rotations that are shorter than optimal. In this case, Faustmann is right that, under the prevailing resource-allocation mechanism, the forest should be valued at the discounted, remaining value. (See Fig. 1.) The difference, \([F(a) + L(a)] e^{-r(a-t)} -\)
\( A(t) e^{rt} - [F(t) + L(a)] > 0 \), can be realized only if the forest is allowed to grow to “maturity” at \( a \).

Depending as it does on the timing of harvest, this difference is the value of an option to harvest at \( a \) rather than at \( t \). The option for this asset undergoing gestation is a part of the value of the single asset.

**Corollary 8** Option. There is an option to cut at any time during the rotation. The option value of the forest at a time before harvest is the difference between two values. One is the value realizable at the chosen harvest date, discounted to \( t \) and net of the flows already obtained. The other is the so-called intrinsic value, the sum of the value of bare land and the net revenue that would be earned from harvesting at \( t \). If the forest is sold as a whole, the value realized includes the option value.

The value of the forest at \( t < a \) is not the net realizable value, but the value (i) inclusive of the option and (ii) net of the cumulative value of any amenities received so far, namely, \( \Phi(a)e^{rt} - A(t)e^{rt} \). This value would be received in a market for the forest with trees of age \( t \). The value of the option at time \( a \) is

\[
\Omega(t, a) = \Phi(a)e^{rt} - A(t)e^{rt} - [R(t) + L(a)]
\]

\[
= \Phi(a)e^{rt} - A(t)e^{rt} - [F(t) - A(t)e^{rt} + L(a)]
\]

\[
= \Phi(a)e^{rt} - [F(t) + L(a)]. \tag{9}
\]

The analysis of harvesting as an option indicates that there is no need for a different analysis for a forest that has age \( t \in (0, \hat{a}) \). (See also Strang 1983.) The optimal harvest age remains \( \hat{a} \), and requires waiting for \( \hat{a} - t \) periods before cutting until the value of the option reaches zero. The decision to wait a period is made for each \( t \in (0, \hat{a}) \) at the time the forest has reached age \( t \).

Even though the decision to harvest is with respect to the age of the trees only, since the land remains invested while the stand is growing, the option value applies
to the composite of stand and land, to the forest. The option value does not apply to
the realizable value of the trees alone. The sum of the realizable values of the trees
and the land is less than the market value:

\[ R(t) + L(a) < R(t) + L(a) + \Omega(t, a). \]

Since the value remaining in the forest (net of flows already realized) is \([\Phi(a) - A(t)]e^{rt}\)
at time \(t < \hat{a}\) and harvesting at \(t\) yields \(R(t) + L(a)\), the following helps to give some
precision to the sometimes vague notion of what capital value is sunk in the forest
during the gestation period.

**Corollary 9** Sunk Value. *The value that is sunk in the forest at any time during a
rotation is the option value.*

Option value arises under certainty and under a non-optimal resource-allocation
mechanism. Discussion of option value is more familiar under optimality. For \(a < \hat{a}\),
the option has a familiar look.

5. **MANAGEMENT AS AN INTANGIBLE ASSET**

Good management consists of maximizing the value of the forest, which is equiva-
lent to maximizing the value of the forest land (finding \(\hat{a}\) to maximize \(L(a)\)) at the
beginning of a rotation. If a different harvest time is chosen, value is lost.

A historical goal of forest management, criticized by Samuelson (1976: 416), has
been maximizing the sustainable yield. Several interpretations can be given to this
objective; for definiteness it is taken herein to mean finding \(\tilde{a}\) such that \(F'(\tilde{a}) =
[F(\tilde{a}) - c]/\tilde{a}. \) (See Fig. 2.) This rotation is longer than optimal: \(\tilde{a} > \hat{a}\). Suppose all
foresters in the industry choose to maximize the sustainable yield as their objective.
In this case, Faustmann’s admonishment to Gehren, that the forest should be valued
at its discounted remaining value from \(\tilde{a}\), is not correct.
Equations (2) through (4) hold for any value of \(a\), not just for \(\hat{a}\) (not just for an optimum). That is to say, if \(\hat{a} = \tilde{a} \neq \hat{a}\), then the value \(L(\tilde{a}) + c = \Phi(\tilde{a})\) (the investment at \(a = 0\)) grows at rate \(r\) to \(L(\hat{a}) + F(\hat{a})\) (total value, given the choice of the rotation period) at age \(\tilde{a}\). The value (and the market price) of bare forest land in the context of the resource-allocation mechanism prescribing the maximum sustained yield is \(L(\tilde{a})\).

The resource-allocation mechanism can be interpreted as being one that seems to most foresters as being incontrovertible. The ability to recognize an option that others do not recognize is an asset. (Think of the legendary figures of securities lore.) If that ability is scarce it commands a positive rent. (If it is not scarce, as is assumed in the analysis of optimal management, its rent is zero.) The option is to choose the most advantageous time to harvest from among a set of alternatives: “Accept the most profitable irreversible investment if and only if its current return exceeds the value of the options thus forfeited” (Bernanke 1983: 90). The decision may be to stop immediately rather than to continue to follow a convention that does not maximize value. In particular, the optimal policy is to wait until \(t = \hat{a}\) if \(t < \hat{a}\) or to cut immediately if \(t > \hat{a}\) (subject to a qualification due to Strang, 1983, if an old-growth forest should not be cut). If \(t > \hat{a}\), the option value if one forester among many recognizes the option for a potential increase in value is

\[
\omega(t, \hat{a}) = [L(\hat{a}) + F(t)] - [L(\tilde{a}) + F(\tilde{a})] e^{-r(\tilde{a} - t)} > 0.
\]

**Corollary 10** Suppose that an old-growth forest would be harvested. For the maximum sustainable yield and other resource-allocation mechanisms that specify rotations that are not optimal, the optimal policy is (i) if the age of the stand is less than the optimal rotation length to wait or (ii) if is greater, to cut immediately. Option value is the difference between the values of the optimal and the specified policies.

In the context of conventional management at the maximum sustainable yield, the
gain can be seen to be the benefit of good management or of entrepreneurship. At any age $t < \bar{a}$ (up to the point of harvest under maximum sustained yield), the shadow value of good management is this option value. This shadow value is labeled “$\sigma$” in Fig. 2. In a perfect market it is paid to the forester with vision.

Vision or entrepreneurship is an intangible or nonmarketed asset. Traditionally such assets have been neglected in capital theory. However, so-called intangible capital is of increasing importance (Abraham 2005). Vision herein is a residual, the difference between good and common management. Faustmann’s formula no. 1 (Theorem 1) applies to both a well managed and a poorly managed forest. The contribution of good management is not independently observable; it can be derived only by a counterfactual analysis. A well managed forest is a single, composite asset comprised of land, stand and management.

**Corollary 11 Intangible Assets.** Intangible assets, such as entrepreneurship, and tangible assets combine to form a single productive asset, the forest. The intangible asset improves results. Total investments in each of the superior and inferior programs earn the prevailing rate of interest.

The fact that Faustmann’s formula applies to a composite, the sum of all investments, may help to explain why capital theory has been able to develop successfully, even though it has neglected intangible investments by subsuming their values into the values of tangible, marketed assets. One device has been use of the internal rate of return and its cousin, the “adjusted” rate of interest, discussed above.

### 6. OLD GROWTH

Let old growth be defined as a state in which the forest is no longer growing. This definition broadens the notion of forest primeval to include a forest that can be grown in a plantation. The analysis herein applies to both. Suppose that the state of old
growth begins at age $\bar{a} > \hat{a}$. Suppose also that the forest is not subject to fire or pestilence. Then, for $t \geq \bar{a}$, $R(t)$ and $\alpha(t)$ are constant at $\bar{R} = R(\bar{a})$ and $\bar{\alpha} = \alpha(\bar{a})$.

A necessary and sufficient condition for not harvesting an old-growth forest, because of its continuing amenity value $\bar{a}$, is that the discounted amenity value be such that

$$\int_0^\infty \bar{a} e^{-rt} dt = \frac{\bar{\alpha}}{r} > \bar{R} + L(\bar{a}).$$

If inequality (10) holds, the forest is not harvested. (For a primeval forest, total amenities are almost surely higher than for a harvested forest that has achieved its maximum size, so that future harvesting is less likely.) In addition, Strang (1988) shows that there is an age $a^* < \bar{a}$ such that, if the current age $t \in (\hat{a}, a^*)$, the forest is harvested immediately and if $t > a^*$ the forest is never harvested.

If inequality (10) does not hold, and if there is no constraint on the rate of harvest, the forest is harvested immediately and the optimal rotation is begun. If the old growth has a perfect substitute that is in elastic supply, the optimal harvest time for all of an old-growth forest is the current date. In such an old-growth forest there are still two natural resources, the stand and the land, but a single asset. There are again joint products from the asset, like wool and sheared sheep. The total value is the sum of the realizable values of the trees and the land. Because the forest is past the optimal harvest age, the option value is zero.

As with an exhaustible resource, when demand is perfectly elastic and stationary and there is no constraint on the rate of cutting, extraction of old growth is immediate. Unlike an exhaustible resource, however, an old-growth forest has a positive opportunity cost of cutting, $\bar{a}/r$. This fundamental feature of old growth provides a disincentive to cut that does not exist for a canonical exhaustible resource. As with other amenity flows, it is attributable to the forest and not to either of land or stand.

Moreover, an old-growth stand is exhaustible but forest land is not. In an exhaustible-resource model the land has no opportunity cost. Old growth is an exhaustible re-
source only if the land has no use once the old-growth stand is harvested. An analogy
is pertinent here. If an immature forest is to be cut for some other use, are the im-
mature trees exhaustible? Let the value of the land in the better use be \( L^* > L(\hat{a}) \).
If
\[
\frac{1}{R(a) + L^*} \frac{d}{da} \left( R(a) + L^* \right) = \frac{1}{R(a) + L^*} \frac{dR(a)}{da} < r,
\]
the indication is to cut immediately. Otherwise one waits until \( \left[ 1/(R(a) + L^*) \right] dR(a)/da = r \), a condition analogous to equation (8).

Old growth may be harvested over time if its demand slopes downward but demand
for new growth is elastic.

- Let \( q_s \in [0, 1] \) represent the fraction of the old growth that is harvested at time
  \( s \) and \( Q_t = \int_0^t q_s ds \in [0, 1] \) represent the cumulative harvest to time \( t \).
- Let the harvest be on the interval \([0, T]\), such that \( Q_T = 1 \).
- Let net cash flow be represented by \( \pi(q_s) \).

The land on which the harvested trees stood is replanted as the forest is harvested
and has value \( q_s L(\hat{a}) \). The problem of the forester responsible for an old-growth
forest at time \( t \in [0, T] \) is to maximize

\[
V(t) = \int_t^T \pi(q_s) e^{-r(s-t)} ds + \int_t^T q_s L(\hat{a}) e^{-r(s-t)} ds + \int_t^T \bar{\alpha} (1 - Q_s) e^{-r(s-t)} ds.
\]

The current-value Hamiltonian for the problem is

\[
H(q, \mu, Q) = \pi(q) + qL(\hat{a}) + \bar{\alpha} (1 - Q) + \mu q.
\]

The Hamiltonian is the sum of the contributions of the two resources, land and stand,
which still form a single asset in keeping with Faustmann’s formula. The optimality
conditions, \( \partial H/\partial q = 0 \) and \( \dot{\mu} = r \mu - \partial H/\partial Q \), imply that, on an optimal path,

\[
\int_0^t \bar{\alpha} e^{r(t-s)} ds = -\mu_t = \pi'(q_t) + L(\hat{a}), t \in [0, T].
\]
The shadow value \( \mu_t \) is composed of contributions from the harvest and the land released by harvest. Moreover, the amenities from old growth affect the evolution of the shadow price,

\[
\frac{\dot{\mu}_t}{\mu_t} = \frac{1}{\pi'(q_t) + L(\hat{a})} \frac{d[\pi'(q_t) + L(\hat{a})]}{dt} = r + \frac{\bar{\alpha}Q_t}{\pi'(q_t) + L(\hat{a})}. \tag{11}
\]

The first equality, for a single unit of the forest, resembles Faustmann’s Formula No. 2 for an optimal forest at the point of harvest, formula (8).

Because of the additional opportunity costs of an old-growth forest, formula (11) differs from Hotelling’s rule for a homogeneous, exhaustible resource, which for shadow value \( \nu_t \) would be

\[
\frac{\dot{\nu}_t}{\nu_t} = \frac{\pi'(q_t)}{\pi'(q_t)} = r.
\]

For an exhaustible resource, \( \bar{\alpha} = 0 \) and \( L(\hat{a}) = 0 \).

The r-percent rule holds that growth in the shadow value of old growth must make up for a negative contribution, the loss of amenity value:

\[
\frac{\dot{\mu}_t}{\mu_t} - \frac{\bar{\alpha}Q_t}{\pi'(q_t) + L(\hat{a})} = r.
\]

**Corollary 12** Faustmann’s Formula No. 2, Old Growth. *Like any forest, an old-growth forest is a single asset that encompasses two resources. Although old growth is exhaustible, an old-growth forest is not an exhaustible resource, because of two additional opportunity costs: Each unit of old growth harvested releases a unit of land and each unit remaining continues to provide amenities. The r-percent rule is a special case of Faustmann’s formula for optimal management. An obvious modification holds if the use of the land after harvest is modified, such as for agriculture or housing development.*
7. CONCLUSION

For either optimal or non-optimal management of a stylized forest, there are two natural resources, stand and land. The combination of those resources is a composite asset that returns the going rate of interest over the chosen exploitation period. Other assets may also be involved; the paper has considered a scarce, intangible, nonmarketed asset, management or entrepreneurship. Since each asset’s internal rate of return is exactly the interest rate, there is, in the forest at least, no need for this concept when all assets are enumerated correctly and comprehensively.

In an optimal or non-optimal program, there is a real option throughout the rotation period. Even though the analysis is under certainty, there is a choice to wait or to strike. In particular, there may be an option to cut early and to obtain the sum of the realizable values of the two assets. The option value applies to the program and is not attributable to any one of the resources that make up the forest. It provides the incentive to the decision maker to make an optimal intertemporal decision.
APPENDIX. RATE OF CHANGE OF TOTAL INTERNALIZED VALUE.

Equation (2) states that
\[ L(a) = \frac{F(a) - c e^{ra}}{e^{ra} - 1}. \]

Simple differentiation w.r.t. \( a \) yields that
\[
(e^{ra} - 1)^2 L'(a) = (F' - r ce^{ra}) (e^{ra} - 1) - re^{ra} (F - ce^{ra})
\]
\[
= F' (e^{ra} - 1) - re^{ra} (F - c)
\]
\[
= 0 \text{ at } \hat{a}.
\]

The derivative of the LHS is
\[
2 (e^{ra} - 1) re^{ra} L' + (e^{ra} - 1)^2 L'' = (e^{ra} - 1)^2 L'' \text{ at } \hat{a}
\]
\[
< 0 \text{ for a unique solution.}
\]

The derivative of the RHS is
\[
F'' (e^{ra} - 1) + r F' e^{ra} - r^2 e^{ra} (F - c) - r F' e^{ra} = F'' (e^{ra} - 1) - r^2 e^{ra} (F - c)
\]
\[
= (e^{ra} - 1)^2 L'' \text{ at } \hat{a}
\]
\[
< 0.
\]

Therefore,
\[
\frac{d}{da} \left( \frac{F'}{F + L} \right) = \frac{F''}{F + L} - \left( \frac{F'}{F + L} \right)^2
\]
\[
= \frac{F''}{F + L} - r^2 \text{ at } \hat{a} \text{ by eq. (7)}
\]
\[
< r^2 \frac{e^{ra}}{e^{ra} - 1} \frac{F - c}{F + L} - r^2
\]
\[
= \frac{r^2 e^{ra}}{(e^{ra} - 1)(F + \hat{L})} \left[ F + \hat{L} - e^{ra} (\hat{L} + c) \right]
\]
\[
= 0 \text{ at } \hat{a} \text{ by eq. (4)}.
\]
REFERENCES


Figure 1. Faustmann's Formula No. 1
Figure 2: Faustmann's Formula No. 2.
Maximum Sustainable Yield at $\hat{a}$ vs. Maximum Present Value at $\hat{a}$. 